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Dynamical symmetries, cyclic and periodic states

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Abstract. We build the exact solution and the instantaneous diagonalization of a quantum system exhibiting a $SU(2)$ or $SU(1, 1)$ dynamical symmetry. A generalized displacement operator determines in both cases the temporal evolution and the diagonalization operators of the system. These operators can be used to characterize the exact and the adiabatical evolution of both periodical and cyclic states whose exact and adiabatical phases can be explicitly found. We consider two examples: a charged particle in the presence of a rotating magnetic field and the degenerate optical parametric oscillator. In particular we present and calculate nontrivial geometrical and dynamical phases for the optical parametric oscillator depending upon the coupling parameter that could indeed be measured using an adequate optical experiment.

1. Introduction

Since Berry's discovery [3] that in a sufficiently slow evolution all initial eigenstates of a quantum system driven by a cyclic time-dependent Hamiltonian must incorporate a nontrivial geometrical phase, the interest of the scientific community on this problem has been steadily increasing. A large body of literature exists which attempts to study the nature of such a phase more closely [22], to connect it with the group-theoretical structure of the system [2, 14] or to generalize it to all types of dynamical journeys (not necessarily of adiabatic nature) [1]. A general method of calculation [17, 18], that can be confronted with the feasible experiments [8, 9], [15] has also been built. However, we believe that in all of these works a set of phenomena certainly related but different in nature are being described whose close connection has not yet been sufficiently clarified. For example, while Berry [4] analysed the behaviour of the instantaneous eigenstates of a physical system in an adiabatical and closed excursion of its Hamiltonian, Aharonov and Anandan [1] considered the cyclical and exact evolution of one of the physical states of the system without requiring even the existence of a Hamiltonian. In this line, Moore and Stedman [17, 18] required additional periodicity conditions for the Hamiltonian. It is obvious that all of these dynamical evolutions are not equivalent and can or cannot occur simultaneously. While the quantal adiabatic theorem guarantees the T -cyclic character of the initial eigenstates of a certain system, the exact evolution of these states does not have to be necessarily closed. It could also happen that given both a system and a certain time of evolution T , no cyclic trajectory in the space of the physical states could be found for this given T and a closed evolution in the sense of Aharonov and Anandan would then be impossible. Therefore a method of characterization of cyclical states [23] for a given Hamiltonian system seems to be necessary. When a dynamical symmetry is known and its exact solution is available, the time evolution

operator can be used for this purpose and the identification of the cyclical (and eventually periodic) states [18] follows as a consequence.

This paper is organized as follows. In sections 1 and 2 we summarize the properties of the Lie algebra $su(2)$ and $su(1, 1)$ which seems necessary to obtain in sections 3 and 5 the instantaneous diagonalization and the exact time evolution of a quantum system with any such dynamical symmetries. The instantaneous eigenstates of the system are unitarily related to the generators of the representation space by means of a generalized displacement operator algebraically determined by the parameters of the system. In section 4 we show that the temporal evolution of this operator is sufficient to determine the quantum phases of the system. The time evolution operator yielding the instantaneous eigenstates is also a displacement operator whose characterization requires us to solve a second-order differential equation which is determined by the parameters of the system. We have also shown that this equation is in fact the classical equation of motion (for systems with a classical analogue). Hence, the classical solution is enough to solve the quantal problem.

In sections 6 and 7 we show how the time evolution operator can be used to characterize the possible cyclical states and show how to calculate the phases appearing by virtue of this transformation. The two cases corresponding to an aperiodical and a periodical Hamiltonian are considered in section 8.

Finally, in sections 9 and 10 we consider two particular systems: a spin j -particle in a rotating magnetic field and the degenerate optical parametric oscillator. The two systems possess closed trajectories in both the parameter space and the physical space (the space of the rays); the corresponding states and its phases are determined. In particular, we have also found that the initial eigenstates of an optical parametric oscillator with a sufficiently low frequency must incorporate in each cycle of evolution of the optical field a Berry phase depending on the intensity of the interaction between the electromagnetic field and the active optical medium.

2. The $su(2)$ and $su(1, 1)$ Lie algebra

We consider three operators $K_0 = K_0^+$ and $K_+^+ = K_-$ verifying the following commutation relations:

$$[K_0, K_{\pm}] = \pm K_{\pm} \quad [K_+, K_-] = 2gK_0. \quad (2.1)$$

When $g = +1$ and $g = -1$ these operators yield a unitary realization of the triparametric algebras $su(2)$ and $su(1, 1)$ respectively. The representation space [25] is generated by simultaneous eigenstates of both K_0 and the quadratic Casimir operator

$$C_2 = K_0^2 + \frac{g}{2}(K_+K_- + K_-K_+) = k(k + g). \quad (2.2)$$

These states are described by two quantum numbers $|k, n\rangle$ and the action of the generators of the algebra on the states is summarized in table 1.

Table 1.

	$su(2)$	$su(1, 1)$
$K_+ k, n\rangle$	$\sqrt{(k-n)(k+n+1)} k, n+1\rangle$	$\sqrt{(n+1)(n+2k)} k, n+1\rangle$
$K_- k, n\rangle$	$\sqrt{(k+n)(k-n+1)} k, n-1\rangle$	$\sqrt{n(n+2k-1)} k, n-1\rangle$
$K_0 k, n\rangle$	$n k, n\rangle$	$(k+n) k, n\rangle$

When $g = 1$ only a set of discrete unitary representations with non-negative values of k , integer or half-integer. The states $|k, n\rangle$ are identified as the states of spin k and projection $n = -k, -k + 1, \dots, 0 \dots k - 1, k$ that are the basis of a unitary and compact representation of $su(2)$ with a finite dimension $2k + 1$.

When $g = -1$ there exist discrete or continuous unitary representations [25]. In particular we shall use a realization of $su(1, 1)$ generated by quadratic products of bosonic operators,

$$K_0 = \frac{1}{2}(a^+a + \frac{1}{2}) \quad K_- = K_+^+ = \frac{1}{2}a^2 \tag{2.3}$$

in which case, k may only have two values $k = \frac{1}{4}$ or $\frac{3}{4}$ and n must be an arbitrary non-negative integer. The states $|k, n\rangle$ are eigenstates of the number operator and can be identified in this realization as the oscillator-even states and oscillator-odd states respectively both generating noncompact unitary representations of infinite dimension

$$|\frac{1}{4}, n\rangle = |2n\rangle \quad |\frac{3}{4}, n\rangle = |2n + 1\rangle. \tag{2.4}$$

3. Factorization of operators

As is well known [7] an element of the group $SU(2)$ can be obtained by exponentiation of an element of the corresponding algebra. It is also well known that we can write down this element in many equivalent factorized ways. This method can also be generalized to describe the fragmentation of a $SU(1, 1)$ elements. The Baker–Hausdorff–Campbell formula allows us to express all unitary elements of $SU(1, 1)$ obtained by exponentiation of an anti-Hermitic element of $su(1, 1)$ as†

$$\begin{aligned} U &= \exp\{\rho K_+ - \rho^* K_- + 2ibK_0\} = \exp\{aK_+\} \exp\{cK_0\} \exp\{dK_-\} \\ &= \exp\{\eta K_+\} \exp\{\gamma K_0\} \exp\{-\eta^* K_-\} \exp\{ihK_0\} \end{aligned} \tag{3.2}$$

where b is a real number and

$$\eta = \frac{\rho}{\Delta} \frac{\tan \Delta}{1 - i(b/\Delta) \tan \Delta} \tag{3.3}$$

$$\gamma = \log\{1 + g|\eta|^2\} \tag{3.4}$$

$$h = 2 \arg \tan \left\{ \frac{b}{\Delta} \tan \Delta \right\} \quad h \in [-\pi, \pi] \tag{3.5}$$

$$\Delta = \sqrt{b^2 + g|\rho|^2}. \tag{3.6}$$

This factorization is valid for $SU(2)$ ($g = 1$) and for $SU(1, 1)$ ($g = -1$). In the latter case this is also possible for all possible values of Δ . For imaginary values of Δ in formulae (3.3)–(3.6) the term $(\Delta^{-1} \tan \Delta)$ becomes $(\Delta'^{-1} \tanh \Delta')$ with $\Delta' = \sqrt{-\Delta^2}$ keeping its real character. Moreover, when $g = -1$, $|\eta| \leq 1$ for any value of ρ and b in such a way that γ is real for all cases too. The correspondence $(\gamma, b) \iff (\eta, h)$ described by formulae (3.3)–(3.6) can thus be inverted. If we characterize the element by the parameters b (real) and $z = \rho/b$ (complex) we always have:

$$z = \eta \frac{e^{-ih/2}}{\sin(h/2)} \tag{3.7}$$

†

$$\exp\{ihK_0\} \exp\{zK_-\} = \exp\{ze^{-ih}K_-\} \exp\{ihK_0\} \tag{3.1}$$

$$b = \frac{1}{\sqrt{1 + g|z|^2}} \arg \tan \left\{ \sqrt{1 + g|z|^2} \tan(h/2) \right\}. \quad (3.8)$$

If $g = 1 \circ g = -1$ and $|z| < 1$ the periodic character of the function in the argument of (3.8) makes the solution for b not unique. Hence, all operators with characteristic parameters (z, b) and (zb', b') being $b' = b + s\pi(1 + g|z|^2)^{-1/2}$ (with s an integer) have the same factorization (η, h) . If $g = -1$ and $|z| \geq 1$ the tangent function becomes a hyperbolic tangent and has a one-to-one correspondence. In particular when one considers $b = 0$ and $\rho = re^{i\phi}$ we recover the well known factorization formulae [11] for the generators of the generalized Perelomov coherent states [19]:

$$S(\rho) = \exp \rho K_+ - \rho^* K_- = \exp \eta K_+ \exp \gamma K_0 \exp -\eta^* K_- = S(\eta) \quad (3.9)$$

$$h = 0 \quad \eta = \tan re^{i\phi} \quad \text{for } SU(2) \quad (3.10)$$

$$h = 0 \quad \eta = \tanh re^{i\phi} \quad \text{for } SU(1, 1). \quad (3.11)$$

In the case of $SU(1, 1)$ the displacement operator $S(\rho) = S(\eta)$ enjoys several interesting properties as opposed to the $SU(2)$ case. This is due—at least mathematically—to the presence in the factorization of the former case of bound functions as the hyperbolic tangent. When the vacuum or a nontrivial coherent state is transformed [21], the quantum noise dispersion properties of these states drastically changes. The fluctuations of the position or momentum operators are reduced under its vacuum value and a new type of quantum states—the squeezed states—are generated. These states represent strictly quantum effects of light with a well-defined amplitude and phase, and with properties which depend on the parameters r and ϕ . Although it is interesting from the group theoretical point of view the case $b = 0$ is just a particular case of factorizable operator. In general for any unitary element of $SU(1, 1)$ with $|\rho| \geq |b|$ one can factorize it with the help of a squeezing operator $S(\beta)$ in the form:

$$U = \exp\{\rho K_+ - \rho^* K_- + 2ibK_0\} = \exp\{\beta K_+ - \beta^* K_-\} \exp\{ihK_0\} \quad (3.12)$$

$$\tanh |\beta| = |\rho| \frac{\sinh \Delta'}{\sqrt{|\rho|^2 \cosh^2 \Delta' - b^2}} \quad (3.13)$$

$$\arg \beta = \arg \rho + \frac{h}{2} \quad (3.14)$$

$$h = 2 \arg \tan \frac{b}{\Delta'} \tanh \Delta' \quad (3.15)$$

$$\Delta' = \sqrt{|\rho|^2 - b^2}. \quad (3.16)$$

Thus all unitary operators of the form (3.12) can be put in the form of a squeezed state operator in the form described above.

4. The instantaneous diagonalization

We shall now consider a physical system described by the Hermitian Hamiltonian

$$H(t) = f(t)K_+ + 2Z(t)K_0 + f^*(t)K_- \quad (4.1)$$

where $f(t) = X(t) + iY(t)$, and $X(t)$, $Y(t)$, $Z(t)$ are time-dependent real functions with arbitrary initial conditions. Using the operators K_0 and K_{\pm} introduced in the previous section, $H(t)$ can be written as a Hermitic element of the Lie algebra $su(2)$ or $su(1, 1)$ depending on the value of g . In order to build the instantaneous eigenstates and the solution of $H(t)$ we shall deal with both cases on the same footing by using the generalized coherent

states of the corresponding symmetry group [19]. According to Perelomov the generalized coherent states can be obtained by applying to the ground state the generalized displacement operator $S(\beta) = S(\eta)$ of the type (3.9). The main idea for constructing (3.9) is to use the Cartan subalgebra of the corresponding symmetry group. The details can be found in [19].

$$S(\eta)K_-S^+(\eta) = \frac{1}{1 + g|\eta|^2} \{K_- + 2g\eta K_0 - g\eta^2 K_+\} \tag{4.2}$$

$$S(\eta)K_0S^+(\eta) = \frac{1}{1 + g|\eta|^2} \{-\eta^* K_- + (1 - g|\eta|^2)K_0 - \eta K_+\} \tag{4.3}$$

$$S(\eta)K_+S^+(\eta) = \frac{1}{1 + g|\eta|^2} \{-g\eta^{*2} K_- + 2g\eta^* K_0 + K_+\}. \tag{4.4}$$

If we consider the parameter $\beta = \beta(t)$ (or $\eta = \eta(t)$) as a time-dependent function we may also consider that the time dependence of $H(t)$ is originated by the action of a time-dependent unitary transformation[†] over an essentially static system. $H(t)$ can then be expressed at each instant of time as:

$$H(t) = 2F(t)S(\eta_0)K_0S^+(\eta_0) \tag{4.5}$$

for any values of $F(t)$ and $\eta_0(t)$ which are algebraically determined by the functions of the Hamiltonian. In particular:

$$F(t) = \sqrt{Z(t)^2 + g|f(t)|^2} \tag{4.6}$$

$$\eta_0(t) = -\frac{f(t)}{Z(t) + F(t)}. \tag{4.7}$$

Therefore, the instantaneous eigenstates of $H(t)$ can be written as $|\eta_0(t), k, n\rangle$

$$|\eta_0(t), k, n\rangle = S(\eta_0(t))|k, n\rangle \tag{4.8}$$

with a time-dependent ‘eigenvalue’ ($\langle K_0 \rangle = \langle k, n | K_0 | k, n \rangle$)

$$E_{k,n}(t) = 2\langle K_0 \rangle \sqrt{Z^2 + g|f|^2} \tag{4.9}$$

which is obtained by means of the action of $S(\eta_0)$ on the stationary eigenstates of K_0 . $F(t)$ is real for any value of the parameters if $g = 1$, but only when $|Z| > |f|$ if $g = -1$. Here an important difference between the dynamical symmetries $SU(2)$ and $SU(1, 1)$ is emphasized. All of the Hermitian elements of the former case are unitarily related to the generator of its Cartan subalgebra. In contrast, just a few elements of the latter case verify this property. Physically, this property has an important consequence. Any physical system whose dynamical system was $su(2)$ could be instantaneously diagonalized in an orthonormal set of vectors obtained by the action of a perfectly identified generalized displacement operator on the purely static eigenbasis of K_0 . Moreover, as this eigenbasis is nondegenerate, this property is preserved. However, when the symmetry is $su(1, 1)$, the existence of this type of eigenvectors can only be defined for times that verify $|Z| > |f|$. This is the necessary and sufficient condition for the physical system under study to have a well-defined fundamental state. Other possibilities cannot be physically acceptable [26].

5. The Berry phase

The physical system described by $H(t)$ depends on time through a set of parameters $\mathbf{R} = \{X(t), Y(t), Z(t)\}$ controlling the external forces or acting on the system. This set may

[†] Not unique.

be interpreted as a vector of coordinates $X(t), Y(t), Z(t)$. As time passes the vector draws a curve in \mathbb{R}^3 : the parameter manifold, described by a set of parametric equations given by $\mathbf{R} = \{X(t), Y(t), Z(t)\}$. We consider in this system a closed or cyclic evolution in the space of the parameters in such a way that after a time T the functions of the system take its initial values again. The Hamiltonian then recovers its initial functional form. When the journey's pace of \mathbf{R} across the parameter space is infinitely slow (the functions $X(t), Y(t)$ y $Z(t)$ are slowly variable functions of the time) the quantum adiabatic theorem [16] guarantees that after the journey the initial eigenstate acquires a phase. This phase adds to the purely dynamical term

$$D_{k,n} = -\frac{1}{\hbar} \int_0^T \langle \eta_0(t), k, n | H(t) | \eta_0(t), k, n \rangle dt = -\frac{2}{\hbar} \langle K_0 \rangle \int_0^T \sqrt{Z^2 + g|f|^2} dt \quad (5.1)$$

another term of geometrical character that is obtained by means of the circulation of a vector [4] along the closed curve which represents the transformation in parameter space. Alternatively, the geometrical contribution can be obtained by evaluating the flux of the rotational of this vector across the surface of this space contained within the above-mentioned closed curve. This vector,

$$\mathbf{v} = \langle n(\mathbf{R}) | \text{Grad}_{\mathbf{R}} | n(\mathbf{R}) \rangle \quad (5.2)$$

can be built using the instantaneous eigenstates of $H(t)$. When the characterization of these states in terms of a displacement operator S is available, a direct calculation of the geometrical phase is in fact both feasible and possible. In fact as the eigenstates can be identified in the form:

$$|n[\mathbf{R}(t)]\rangle = |\eta_0(t), k, n\rangle \quad (5.3)$$

and since [5]:

$$S^+(\eta) \dot{S}(\eta) = \frac{\dot{\eta}}{1 + g|\eta|^2} K_+ - g \frac{\dot{\eta}\eta^* - \eta\dot{\eta}^*}{1 + g|\eta|^2} K_0 - \frac{\dot{\eta}^*}{1 + g|\eta|^2} K_- \quad (5.4)$$

we finally obtain the Berry phase as:

$$B_{k,n} = \langle K_0 \rangle \int_0^T \left(1 - \frac{Z}{\sqrt{Z^2 + g|f|^2}} \right) \frac{d(\arg(f))}{dt} dt. \quad (5.5)$$

The existence of a nontrivial Berry phase requires that the argument of the complex parameter which characterizes the time-dependent system (with dynamical symmetry $SU(2)$ or $SU(1, 1)$) must indeed depend on time. For a set of nondegenerate states such as those we have used here, one can define a two-form. This two-form represents the flux across the surface in parameter space and also equals the magnitude of the Berry phase. It can be extracted from:

$$\text{Rot} \mathbf{v} = g \frac{\mathbf{R}}{\sqrt{Z^2 + g(|X|^2 + |Y|^2)}^3} \quad (5.6)$$

$$B_{k,n} = g \langle K_0 \rangle \int \int_S \frac{\mathbf{R} d\mathbf{S}}{\sqrt{Z^2 + g(|X|^2 + |Y|^2)}^3}. \quad (5.7)$$

Experimental measures of the Berry phase [4, 12], usually compare the state of the system transported by the Hamiltonian $H(t)$ with the same state transported by the initial Hamiltonian. The superposition of such a state gives a direct measure of the Berry phase. Restricting the parameters space to the surface of \mathbb{R}^3 given by

$$Z^2 + g(X^2 + Y^2) = R^2 = \text{constant} \quad (5.8)$$

and the possible trajectory to any curve C on this surface, the Berry phase is obtained by means of the flux of the $\text{Rot}v$ across the part of S encircled by C .

When $g = 1$, the surface is a sphere of radius R and the Berry phase is reduced to the well known result. The phase is a measure of the solid angle subtended by the closed curve C on the surface, described by the parameter vector \mathbf{R} :

$$B_{k,n} = n \int_{\Omega} d\Omega. \quad (5.9)$$

When $g = -1$, the parameter space can be visualized as the upper part of a two-sheeted hyperboloid and the Berry phase can no longer be interpreted in terms of a solid angle. However, the procedure used to obtain the value of the phase is quite similar to the previous one. We have:

$$B_{k,n} = -\frac{n + \frac{1}{2}}{2R^3} \int_S \mathbf{R} dS. \quad (5.10)$$

6. Time evolution operator

The time evolution operator $U(t)$ corresponding to $H(t)$ verifies the Schrödinger equation and reduces itself to the identity for $t = 0$. It must be found among the unitary elements of the symmetry group. This element can be built by exponentiation of a Hermitian element of the algebra and can also be factorized as follows

$$U(t) = \exp\{\beta(t)K_+ - \beta^*(t)K_-\} \exp\{ih(t)K_0\} \quad (6.1)$$

where $h(t)$ and $\beta(t)$ are respectively a real and complex function of time that must be determined by the Schrödinger equation with the initial condition $h(0) = \beta(0) = 0$. It has been shown [5] that the Schrödinger equation is totally equivalent to the nonlinear Riccati equation which is satisfied for the complex function $\eta(t)$.

$$\dot{\eta} = \frac{1}{i\hbar}(f + 2Z\eta - gf^*\eta^2) \quad \eta(0) = 0. \quad (6.2)$$

After solving this equation, its solution $\eta(t)$ gives $\beta(t)$ and $h(t)$ after some trivial quadratures. This completes the procedure to uniquely obtain the time evolution of the physical system

$$h(t) = -\frac{1}{\hbar} \int_0^t \{2Z(s) - g[\eta(s)f^*(s) + f(s)\eta^*(s)]\} ds \quad (6.3)$$

$$\beta(t) = \arg \left\{ \tanh(\sqrt{-g}|\eta|) \right\} e^{i\arg(\eta)}. \quad (6.4)$$

The nonlinear Riccati equation can be easily linearized by means of the following change of variables:

$$\eta(t) = g \frac{i\hbar \dot{q}(t)}{f^* q(t)} + \frac{g}{f^*} \left(Z + i\hbar \frac{f^*}{2f^*} \right) \quad (6.5)$$

$$h(t) = \{2 \arg[q(t)] - 2 \arg[q(0)] - \arg[f(t)] + \arg[f(0)]\} \quad (6.6)$$

that reduces the problem to finding a complex function $q(t)$ which must in turn be the solution of the following second-order linear differential equation:

$$\ddot{q}(t) + \Omega^2(t)q(t) = 0 \quad (6.7)$$

$$\frac{\dot{q}(0)}{q(0)} = \frac{i}{\hbar} Z(0) - \frac{f^*(0)}{2f^*(0)} \quad (6.8)$$

with the following time-dependent complex frequency

$$\Omega^2(t) = \frac{1}{\hbar^2}(Z^2 + g|f|^2) - \frac{i}{\hbar} \left(\dot{Z} - Z \frac{\dot{f}^*}{f^*} \right) + \frac{\ddot{f}^*}{2f^*} - \frac{3\dot{f}^{*2}}{4f^{*2}}. \quad (6.9)$$

Usually one should try to find this solution through a linear superposition of two independent particular solutions of (6.7). Alternatively, one can consider that two exact copies of (6.7) are in fact an Ermakov differential equation system [20]. These classical systems enjoy the property of always possessing an invariant. Therefore one can conclude that only one solution becomes necessary. We can build the other one by means of a quadrature. Let $q_0(t)$ be a particular solution of (6.7). The linear superposition

$$q(t) = Aq_0(t) + Bq_0(t) \int \frac{dt}{q_0^2(t)} \quad (6.10)$$

depends on two arbitrary constants A and B . It is the general solution of (6.7).

When $g = -1$, there exists an alternative linear system associated to the Riccati equation (6.2) that is obtained with the following homogeneous change of variables

$$\eta(t) = \frac{1 - \xi(t)}{1 + \xi(t)} \quad (6.11)$$

$$\xi(t) = i \left(\frac{\hbar}{Z_-} \frac{\dot{q}(t)}{q(t)} - \frac{\text{Im}(f)}{Z_-} + \frac{\hbar \dot{Z}_-}{2Z_-^2} \right) \quad (6.12)$$

where $Z_- = \text{Re}(f) - Z$. The frequency of the linear equivalent equation (6.7) reads:

$$\Omega^2(t) = \frac{1}{\hbar^2}(Z^2 - |f|^2) + \frac{1}{\hbar} \left(\text{Im}(f) \frac{\dot{Z}_-}{Z_-} - \frac{d}{dt} \text{Im}(f) \right) + \frac{\ddot{Z}_-}{2Z_-} - \frac{3\dot{Z}_-^2}{4Z_-^2}. \quad (6.13)$$

This frequency is now real and it can be shown that for those systems with a classical analogue such as the generalized harmonic oscillator [5] or the conformal oscillator the equation (6.7) with frequency given by (6.13) is in fact the classical equation of motion. Thus this procedure describes a reduction process that allows one to build the exact solution to the quantum problem in terms of the exact solution of the equivalent classical problem. More precisely, the exact evolution of any quantum state of a system with $SU(1, 1)$ dynamical symmetry and also possessing a classical analogue is determined by only one real classical solution. Nothing of this nature can be found for the $SU(2)$ case.

7. Characterization of the cyclic states

Let us now consider the process for which a given state of Hilbert space acquires a phase after some time T when evolving in a naturally driven way by $U(t)$ (i.e. with no adiabatic assumption involved). The evolution can be represented as:

$$|\Psi_0\rangle \implies U(t)|\Psi_0\rangle \implies U(T)|\Psi_0\rangle = e^{i\varphi}|\Psi_0\rangle. \quad (7.1)$$

It is important to emphasize at this point that the existence of these state vectors (hereafter called T -cyclic) is not guaranteed for a given system and/or a given time T . Even when they do exist there is nothing in the theory that forces them to be state vectors of the initial Hamiltonian. The above-mentioned existence of this kind of state vector can be established with the previous knowledge of $U(T)$ and the following additional properties.

- (i) A state vector can be said to be T -cyclic if and only if it is an eigenstate of $U(T)$.
- (ii) If $U(t)$ can be written as:

$$U(t) = \exp\{iM(t)\} \quad M(t) = M^+(t) \quad (7.2)$$

any eigenvector $|\Phi_M(T)\rangle$ of $M(T)$ is a T -cyclic state vector of the system.

(iii) After an interaction time T , a T -cyclic state vector acquires a phase $\exp\{i\lambda_M(T)\}$ that is obviously given through the set of eigenvalues $\lambda_M(T)$ of $M(T)$.

For a physical system with a $SU(2)$ or $SU(1, 1)$ dynamical symmetry and for which the Riccati equation (6.2) has been solved for any of the already described methods, the functions $\eta(t)$ and $h(t)$ should be known and the results of section 3 allows us to write $M(T)$ as:

$$M(T) = -ib(T)\{x(T)K_+ - x^*(T)K_- + 2iK_0\} \tag{7.3}$$

$$x(T) = \frac{\eta(T)e^{-ih(T)/2}}{\sin(h(T)/2)} \tag{7.4}$$

$$b(T) = \frac{1}{\Delta(T)} \arg \tan\{\Delta(T) \tan(h(T)/2)\} \tag{7.5}$$

$$\Delta(T) = \sqrt{1 + g|x(T)|^2}. \tag{7.6}$$

The eigenvectors $|\Phi_M(T)\rangle$ of $M(T)$ are related to those of K_0 through a displacement operator $S(\rho_0(T))$ of the form:

$$|\Phi_M(T)\rangle = S(\rho_0(T))|k, n\rangle \tag{7.7}$$

where $\rho_0(T)$ is given as in (4.7):

$$\rho_0(T) = i \frac{x(T)}{1 + \Delta(T)}. \tag{7.8}$$

The corresponding eigenvalues are

$$\lambda_M(T) = 2\langle K_0 \rangle \omega_T T \tag{7.9}$$

$$\omega_T = \frac{1}{T} \arg \tan \left\{ \Delta(T) \tan \left(\frac{h(T)}{2} \right) \right\}. \tag{7.10}$$

If we now consider the instantaneous diagonalization properties they appear to be quite different for the $su(2)$ and $su(1, 1)$ cases. For $su(2)$ any element of the form (7.3) is always diagonalizable. This means that if an arbitrary time interval of time T is kept fixed, it is always possible to find an orthonormalized collection of T -cyclic states. If the system has been prepared in one of these states ((7.7) for $g = 1$) it will return to the same state after a time T but having an extra phase given by (7.9). Moreover, the time evolution operator can always be written in the form:

$$U(t) = \exp i\{2\omega_T t S[\rho_0(t)] K_0 S^+[\rho_0(t)]\}. \tag{7.11}$$

However, for the $su(1, 1)$ case the above properties do not generally hold. There only exist cyclic states for those values of the time T with $|x(T)| > 1$ and the characterization (7.11) for $U(t)$ is not generally true for all instants of time. If T is one of these permitted values the possible cyclic states and its phases are given by (7.7)–(7.10) with $g = -1$. It is important to emphasize that in both cases there is no reason for these T -cyclic states to come back to the initial situation in a periodic manner. They may come back once (T -cyclic) but not twice or more repeatedly (T -periodic). We do not believe that this distinction has been made yet in the current literature of the subject.

8. Characterization of cyclic states in periodic systems

The process described in the last section is independent of the periodicity of the Hamiltonian. The T -cyclic states will be T -periodic too whenever solutions $x(t)$ exist that take the same value after a time T , regardless of whether the solutions $x(t)$ are periodic or not themselves

$$x(sT) = x(T) \quad s = \text{integer} \geq 1. \quad (8.1)$$

If we now impose to $H(t)$ an additional periodicity condition [17, 18] there is in principle no reason to assume that the cyclic states are just restricted to those whose characteristic time of cyclicity coincides with the period of $H(t)$. Only the procedure described above may uniquely characterize the T -cyclic states.

If the system has additionally T -periodic states its period again may or may not coincide with the period of the Hamiltonian $H(t)$. To identify these states we can use another factorization of $U(t)$ different from the one given in (7.2). The Floquet theorem—a version of the Bloch theorem that classifies the eigenfunctions of periodic potentials—establishes that in a Hamiltonian system of periodicity T , $U(t)$ can be factorized not uniquely [17, 18] by means of a unitary operator with the same periodicity of the Hamiltonian and a constant operator. Both are determined by the following time evolution operator:

$$U(t) = Z(t) \exp\{i\tilde{M}t\} \quad Z(t+T) = Z(t) \quad (8.2)$$

$$\tilde{M} = -\frac{i}{T} \log U(T) \quad Z(t) = U(t) \exp\{-i\tilde{M}t\}. \quad (8.3)$$

The periodic states of the system are the eigenstates of a constant operator \tilde{M} that can be constructed in terms of the above as $\tilde{M} = \frac{M(T)}{T}$. In turn $M(T)$ is given in (7.3) and must now be obtained by means of a function $x(t)$ built with the help of a solution $\eta(t)$ of the Riccati equation (6.2) that now has periodical coefficients of period T . It is important to point out again that there is no reason for $\eta(t)$ to be a periodic function. The periodic operator $Z(t)$ can also be built by using this function $\eta(t)$ which—as we have already said—will not be, in general, periodic. Using the factorization formulae that we have used frequently in this paper, we can write

$$\exp\{-i\tilde{M}t\} = S[\eta'(t)] \exp\{-ih'(t)K_0\} \quad (8.4)$$

and characterize in an explicit manner† $Z(t)$ in various different although equivalent ways as:

$$Z(t) = S(\eta) \exp\{ihK_0\} S(\eta') \exp\{-ih'K_0\} = S(\eta'') \exp\{ih''K_0\} \quad (8.5)$$

where

$$\eta'(t) = -x(T) \frac{\tan(\omega_T t)}{\Delta(T) + i \tan(\omega_T t)} \quad (8.6)$$

$$\eta''(t) = \frac{\eta(t) + \eta'(t)e^{ih(t)}}{1 - g\eta^*(t)\eta'(t)e^{ih(t)}} \quad (8.7)$$

$$h'(t) = 2 \arg \tan \left\{ \frac{1}{\Delta(T)} \tan(\omega_T t) \right\} \quad (8.8)$$

$$h''(t) = h(t) - h'(t) + i \log \frac{1 - g\eta^*(t)\eta'(t)e^{ih(t)}}{1 - g\eta(t)\eta'^*(t)e^{-ih(t)}}. \quad (8.9)$$

†

$$S(\eta_1)S(\eta_2) = S(\eta) \exp\{ihK_0\} \quad \text{with } \eta = \frac{\eta_1 + \eta_2}{1 - g\eta_1^*\eta_2} \quad \text{and} \quad h = i \log \left\{ \frac{1 - g\eta_1^*\eta_2}{1 - g\eta_1\eta_2^*} \right\}.$$

9. Calculation of the geometrical phase

If the system is initially prepared in one of the T -cyclic states given by (7.7) the natural evolution of these states yields after a time T a phase factor of the general form $\exp\{i\lambda_M(T)\}$ with $\lambda_M(T)$ given in (7.9). The instantaneous form of these states depends on the action of $U(t)$ and can be written exactly as:

$$\begin{aligned} |\Phi_M(t)\rangle &= U(t)|\Phi_M(T)\rangle = S[\eta(t)] \exp\{ihK_0\} S[\rho_0(T)]|k, n\rangle \\ &= \exp\{i\delta(t)\langle K_0\rangle\} S[\rho(t)]|k, n\rangle \end{aligned} \quad (9.1)$$

where

$$\rho(t) = \frac{\eta(t) + \rho_0(T)e^{ih(t)}}{1 - g\eta^*(t)\rho_0(T)e^{ih(t)}} \quad (9.2)$$

$$\delta(t) = h(t) + i \log \frac{1 - g\eta^*(t)\rho_0(T)e^{ih(t)}}{1 - g\eta(t)\rho_0^*(T)e^{-ih(t)}} \quad (9.3)$$

whose average 'energy' is given by

$$\langle \Phi_M(t) | H(t) | \Phi_M(t) \rangle = 2 \frac{\langle K_0 \rangle}{1 + g|\rho|^2} \{ (1 - g|\rho|^2)Z - g(f\rho^* + f^*\rho) \} \quad (9.4)$$

and so, the dynamical part of the phase takes the form:

$$D_{k,n}(T) = -2 \frac{\langle K_0 \rangle}{\hbar} \int_0^T \frac{[1 - g|\rho(t)|^2]Z(t) - g[f(t)\rho^*(t) + f^*(t)\rho(t)]}{1 + g|\rho(t)|^2} dt \quad (9.5)$$

and its geometrical part may be found by subtracting the dynamical phase from the total phase: $\beta = \lambda_M(T) - D_{k,n}(T)$. In the above case we were dealing with a periodic Hamiltonian and its periodic states this phase can be obtained alternatively by using the time variation of $Z(t)$ [18]:

$$\beta = i \int_0^T \langle \Phi_M(T) | Z^+(t) \dot{Z}(t) | \Phi_M(T) \rangle. \quad (9.6)$$

10. Example I: spin j in a magnetic field

We shall now consider two examples in which it is possible to apply the above-mentioned formalism to characterize and construct cyclical and/or periodical states and their phases and to study the two types of closed transformations, i.e. in parameter space and in Hilbert space. First we shall consider a spin j -particle imbedded in a magnetic field which precesses around the z -axis with an angular frequency ω

$$\mathbf{B} = (B \cos \omega t, B \sin \omega t, B_0). \quad (10.1)$$

The dynamical symmetry of this system is $su(2)$ and the characteristic functions are [10]:

$$f = -\frac{\hbar}{2} b \exp^{-i\omega t} \quad Z = -\frac{\hbar}{2} b_0 \quad (10.2)$$

where we have called $\mu B = b$ and $\mu B_0 = b_0$. With these definitions we proceed to solve in order, the following steps.

- *Riccati equation.*

$$\eta(t) = i \frac{b}{b_1} \sin\{\phi(t)\} \exp^{i[\phi(t) - \omega t]} \quad (10.3)$$

$$h(t) = -\omega t + 2\phi(t) \quad (10.4)$$

where $b_1 = b_0 + \omega$, $\omega_c = \sqrt{b^2 + b_1^2}$ and $\phi(t)$ is a time-dependent function defined by the equation:

$$\tan\{\phi(t)\} = \frac{b_1}{\omega_c} \tan\left(\frac{\omega_c t}{2}\right). \quad (10.5)$$

• *Instantaneous eigenstates and Berry phases.* The parameter space is the sphere of radius $\sqrt{b^2 + b_0^2}$ determined by the magnetic field \mathbf{B} . When this magnetic field precesses around Z draws the circumference defined by the intersection of the sphere with the plane $Z = b_0$. For an adiabatic precession of \mathbf{B} the transformation can be considered adiabatic and the instantaneous eigenstates:

$$\eta_0(t) = -2 \frac{b \exp^{-i\omega t}}{b_0 + \sqrt{b^2 + b_0^2}} \quad (10.6)$$

have a Berry phase of the form:

$$B_m = \int_0^T \left(1 - \frac{Z}{\sqrt{Z^2 + |f|^2}}\right) \frac{d(\arg(f))}{dt} dt = -2m\pi \left(1 - \frac{b_0}{\sqrt{b^2 + b_0^2}}\right) \quad (10.7)$$

that, as is described in [13], is m times the solid angle subtended from the origin by the circumference defined above.

• *Cyclic states.* When a closed evolution in the Hilbert space is considered we shall find the cyclic states of the system. For any value of the time T the orthonormalized collection of $2j + 1$ states

$$|\phi_m(T)\rangle = S[\rho_0(T)]|j, m\rangle \quad (10.8)$$

$$\rho_0(T) = i \frac{x(T)}{1 + \Delta(T)} \quad (10.9)$$

$$x(T) = i \frac{b}{b_1} \frac{\sin[\phi(T)]}{\sin[\phi(T) - \omega T/2]} e^{-i\frac{\omega T}{2}} \quad (10.10)$$

returns T units of time after the interaction has been turned on. However, after the journey they carry out a new phase of the form:

$$\lambda_m(T) = 2m \arg \tan\{\Delta_T \tan[\phi(T) - \omega T/2]\} \quad (10.11)$$

whose geometrical and nonadiabatic contribution can be obtained by subtracting the exact dynamical part:

$$D_m(T) = \frac{m}{\omega_c^2 \Delta_T} [b_0 b_1 + b^2][b_1 + b \epsilon_T \cos(\omega T/2)] T - \frac{m \omega b}{\omega_c^2 \Delta_T} \times \left\{ [b - b_1 \epsilon_T \cos(\omega T/2)] \frac{\sin(\omega_c T)}{\omega_c} - \epsilon_T \sin(\omega T/2)[1 - \cos(\omega_c T)] \right\} \quad (10.12)$$

where we have called:

$$\Delta_T = \sqrt{1 + |x(T)|^2} \quad (10.13)$$

$$\epsilon_T = \frac{b}{b_1} \frac{\sin[\phi(T)]}{\sin[\phi(T) - \omega T/2]}. \quad (10.14)$$

• *Periodical states.* The system is driven by a periodical Hamiltonian with a period $T = \frac{2\pi}{\omega}$ but as $\eta(t)$ is not a periodic function $U(t)$ cannot be reduced in the standard way. Therefore in every cycle of the exact evolution of an eigenstate of K_0 this eigenstate does

not return to the same state. Also $x(t)$ is not periodic but one can trivially show that it verifies:

$$x\left(2s\frac{\pi}{\omega}\right) = i\frac{b}{b_1} \quad s, \text{ integer} \quad (10.15)$$

and the system possesses some $\frac{2\pi}{\omega}$ -periodic states. Let us find them.

First we must realize that these states must be the $S(-\frac{b}{b_1+\omega_c})|j, m\rangle$ eigenstates of the operator

$$M = \frac{b}{2}(K_+ + K_-) + b_1 K_0 \quad (10.16)$$

that in every cycle of evolution of $H(t)$ add global (dynamical and geometrical) phases

$$\lambda_m\left(2\frac{\pi}{\omega}\right) = 2m\pi\left(\frac{\omega_c}{\omega} + 1\right) \quad (10.17)$$

$$D_m\left(2\frac{\pi}{\omega}\right) = 2m\pi\frac{b_0b_1 + b^2}{\omega\omega_c} \quad (10.18)$$

$$\beta_m\left(2\frac{\pi}{\omega}\right) = 2m\pi\left(\frac{b_1}{\omega_c} + 1\right). \quad (10.19)$$

These states are equivalent to those described in [10] in the case of an electron. However, they are not the only periodic states that may be considered. The nonperiodic function $\eta(t)$ also verifies the property

$$\eta\left(2\frac{\pi}{\omega_c}\right) = \eta(0) = 0 \quad s, \text{ integer} \quad (10.20)$$

and the time evolution operator cyclically transports the eigenstates of K_0 with dynamical and geometrical phases given by:

$$\lambda_m\left(2\frac{\pi}{\omega_c}\right) = -2m\pi\left(\frac{\omega}{\omega_c} + 1\right) \quad (10.21)$$

$$D_m\left(2\frac{\pi}{\omega_c}\right) = 2m\pi\frac{b_1(b_0b_1 + b^2)}{\omega_c^3} \quad (10.22)$$

$$\beta_m\left(2\frac{\pi}{\omega_c}\right) = -2m\pi\left(\frac{\omega}{\omega_c} + \frac{b_1(b_0b_1 + b^2)}{\omega_c^3} + 1\right). \quad (10.23)$$

11. Example II: the degenerate optical parametric oscillator

The Hamiltonian of the system is well known [11, 24] and may be identified as a Hermitian element of the $su(1, 1)$ algebra with the notation of section 4 and with characteristic parameters:

$$f(t) = 2\hbar\omega_0\kappa \exp^{-2i\omega_0 t} \quad Z(t) = \hbar\omega_0. \quad (11.1)$$

The $e^{\pm 2i\omega_0 t}$ factors represent the activation mode which will be treated as a classical external field and κ is the real coupling constant.

- *Riccati equation.* Its exact solution is:

$$\eta(t) = -i \tanh(2\kappa\omega_0 t) \exp^{-2i\omega_0 t} \quad h(t) = -2\omega_0 t. \quad (11.2)$$

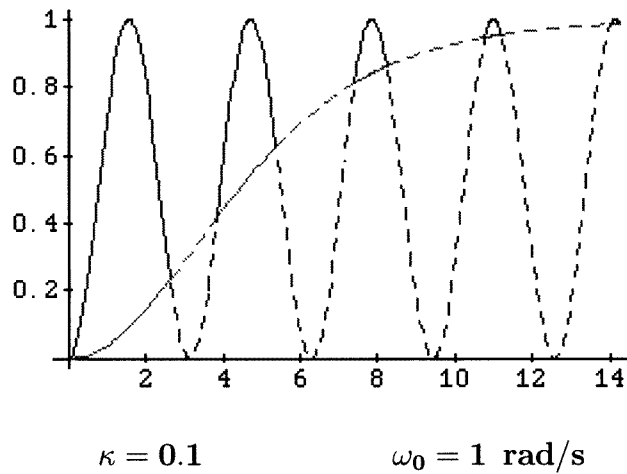


Figure 1.

- *Cyclic states.* For this system one can easily find:

$$x(T) = i \frac{\tanh(2\kappa\omega_0 T)}{\sin(\omega_0 T)} e^{-i\omega_0 T} \quad (11.3)$$

$$\Delta(T) = \sqrt{1 - \frac{\tanh^2(2\kappa\omega_0 T)}{\sin^2(\omega_0 T)}}. \quad (11.4)$$

From this one can deduce that there exist only T -cyclic states with characteristic time T defined by the inequality:

$$\tanh^2(2\kappa\omega_0 T) < \sin^2(\omega_0 T) \quad (11.5)$$

that is represented in figure 1 for a particular value of the parameters. A variation in the choice of the parameters does not affect qualitatively the following conclusions. The distribution of the allowed times defining T -cyclic states clearly reminds one of the spectrum of a particle in a periodic potential and its band structure. This is largely so owing to the periodic nature of the time-dependent Hamiltonian $H(t)$. The similarity becomes more evident if we consider the classical analogue of our system whose classical equations of motion are:

$$\ddot{q}(t) + \Omega^2(t)q(t) = 0 \quad (11.6)$$

$$\Omega^2(t) = \omega_0^2 \frac{1 - 24\kappa^2 + 4\kappa(1 + 8\kappa^2)\cos(2\omega_0 t) - 16\kappa^4\cos^2(2\omega_0 t)}{[1 - 2\kappa\cos(2\omega_0 t)]^2} \quad (11.7)$$

which shows an obvious correspondence with the Schrödinger equation for a periodic time-dependent potential. For T , one of the allowed times, the T -cyclic state vectors of the system are:

$$|\phi_n(T)\rangle = S[\rho_0(T)]|n\rangle \quad (11.8)$$

which, as has already been established, are unitarily related to the eigenstates of the oscillator $|n\rangle$ by means of the displacement operator (7.7) with $\rho_0(T)$ given by (7.8) and $x(T)$ and $\Delta(T)$ given by (11.3) and (11.4) respectively. After an elapsed time T the state acquires a

total phase $\lambda_n(T)$ which is the sum of a dynamical phase $D_n(T)$ and a geometrical phase $\beta_n(T)$ all three being given by the following expressions:

$$\lambda_n(T) = -(n + \frac{1}{2}) \arg \tan \left\{ \sqrt{1 + |x(T)|^2} \tan(\omega_0 T) \right\} \quad (11.9)$$

$$D_n(T) = -\frac{n + \frac{1}{2}}{2\kappa \Delta_T} \tanh(2\kappa \omega_0 T) \left\{ 1 - \frac{4\kappa^2 \omega_0 T}{\tan(\omega_0 T)} \right\} \quad (11.10)$$

$$\beta_n(T) = \lambda_n(T) - D_n(T). \quad (11.11)$$

• *Periodic states.* In our case the Hamiltonian is periodic with period $T = \frac{\pi}{\omega_0}$, but as $\eta(t)$ is not a periodic function, $U(t)$ cannot be reduced to a phase operator for all possible values of s . Thus after each cycle of the exact evolution of a given state with a well-defined photon number such a state does not come back over itself. Furthermore, as for this particular value of T we fall into a forbidden zone (see figure 1), we can safely conclude that the degenerate optical parametric oscillator has no $\frac{\pi}{\omega_0}$ -periodic states. On the other hand, equation (8.1) does not hold for $x(t)$ of (11.3) for any value of $s > 1$. Hence each state comes back over itself just once.

• *Instantaneous eigenstates.* When $\kappa < 0.5$ the system possesses a set of instantaneous eigenstates with a characteristic parameter given by:

$$\eta_0 = -\frac{2\kappa}{1 + \sqrt{1 - 4\kappa^2}} \exp^{-2i\omega_0 t} \quad (11.12)$$

that for sufficiently low frequencies (i.e. guaranteeing an adiabatic evolution of the system) acquire after a time $\frac{\pi}{\omega_0}$ a phase whose dynamical and geometrical (Berry) contributions depend upon the characteristic parameter and turns out to take the following explicit form:

$$D_n = -\pi(n + \frac{1}{2})\sqrt{1 - 4\kappa^2} \quad (11.13)$$

$$B_n = \pi(n + \frac{1}{2})\frac{1 - \sqrt{1 - 4\kappa^2}}{\sqrt{1 - 4\kappa^2}}. \quad (11.14)$$

The value of B_n equals the flux of the vector

$$X(t) = \text{Re } f(t) \quad Y(t) = \text{Im } f(t) \quad Z(t) = \hbar\omega_0 \quad (11.15)$$

through the portion of the hyperboloid

$$Z^2 - X^2 - Y^2 = R^2 \quad (11.16)$$

bounded between its tip and the intersection plane $Z(t) = \hbar\omega_0$. This plane cuts the hyperboloid by drawing a circle in the (X, Y) -plane of radius $R = \hbar\omega_0\sqrt{1 - 4\kappa^2}$. This is reminiscent of the $SU(2)$ case in which the Berry phase is given by the flux through the solid angle as it has been discussed before.

We refer the interested reader to a thorough discussion of these and other properties of the phases in the case of the degenerate optical parametric oscillator which also contemplates some possible experimental consequences [6].

12. Conclusions

In this paper we have discussed the conditions under which any physical system possessing a dynamical symmetry can be instantaneously diagonalized and a well-defined set of orthonormal eigenvectors can or cannot be obtained by the action of a generalized displacement operator acting on the purely static eigenbasis of K_0 : the one-parameter Cartan subalgebra. The answer is that this can always be done if the dynamical symmetry is

governed by the Lie algebra of $su(2)$. However, if the dynamical symmetry is determined by the Lie algebra of $su(1, 1)$ the existence of this type of eigenvectors cannot always be guaranteed. One needs to study whether the parameters of the Hamiltonian impose additional conditions which have to be fulfilled for this eigenbasis to exist. We have hereby given a general formalism for finding such a set of additional conditions.

The time evolution operator that yields this set of instantaneous eigenstates can also be built in terms of a displacement operator whose characterization requires us to solve a nonlinear riccati equation. When we linearize this equation one generally finds a harmonic oscillator with a complex time-dependent frequency. In the case $su(1, 1)$, this frequency is real and we end up simply with the classical equations of motion in the case where the system would have a classical analogue. The quantum version of the problem may also be solved just by using the classical solutions. After having done this we also show that the time evolution operator can be used to characterize the possible cyclic states, the periodic states and the necessary interaction time for making the evolution of the system either cyclic and/or periodic. This discussion allows one to define the exact (as opposed to adiabatical) phases that this transformation yields. It has been found in particular that for $su(2)$ and by fixing an arbitrary interval of time T it is always possible to find an orthonormal set of T -cyclic states. For $su(1, 1)$, however, this is not generally true. T -cyclic states only exist for those values of time T verifying certain conditions. A similar analysis has been carried out for the case of T -periodic physical systems.

Finally we have considered two particular and interesting examples: a spin j -particle in a rotating magnetic field and the degenerate optical parametric oscillator. We have built in both cases its cyclic and periodic states and the dynamical and geometric phases have been calculated. In particular we have found that the initial eigenstates of a degenerate optical parametric oscillator with sufficiently low frequency must incorporate in each cycle of evolution of the optical field a nontrivial Berry phase depending upon the interaction coupling constant between the electromagnetic field and the medium.

We believe that the discussions and results presented in this paper will be of interest to elucidate the differences between adiabatical, cyclic and periodic states of a time-dependent physical system. These differences are shown to be not just of conceptual origin but also giving rise to exactly calculable phases which differ not only in origin but also in numerical values. These and other predictions can now be tested experimentally either in optical Hamiltonians or in spin systems of various kinds. The advantage of having a useful and unified formalism to deal with at the same time as all of these different phenomena is not the smallest contribution of this paper.

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